An e-function nomogram, with special reference to the determination of equilibrium-moisture content and capillary conductivity of soil samples

PH. TH. STOL

Institute for Land and Water Management Research, Wageningen, Netherlands

Summary

The present paper describes a nomogram consisting of a family of curves of the exponential type; it may be used for determining in a simple manner the relation between experimental data and time, when a function as given in formula (6) can be expected to hold. In our case this nomogram was used to determine the physical relation between the decrease of the moisture content in a soil sample and the time during which a constant suction force is applied, in other words the moisture-retention or pF-curve and the capillary conductivity of a soil sample.

The advantage of using the nomogram is that the results of all weighings made for the purpose of finding the equilibrium-moisture content at a certain suction, are used for determining the asymptotical equilibrium value. This greatly increases the accuracy of the determination of that value. If the moisture content still decreases after several days, as is for example often the case at pF 2.7, the equilibrium-moisture content can be safely determined by means of extrapolation. The time required for the experiment can thus be considerably reduced. The graphical representation of the course of the measurements also gives a direct indication of the occurrence of experimental errors.

It is also possible to judge when and at what intervals the weighings should be made in order to obtain the highest accuracy of the determination, since this depends on the capillary conductivity of the sample, which can be read from the slope of the curves made. At low values of the time t, *i.e.* shortly after applying a new suction force, the curves show a steep gradient. An observation at this period of the experiment will mainly determine to what extent the nomogram should be shifted in the direction of the t co-ordinate. The moment of weighing and the interval between weighings are therefore very important at the beginning of the experiment.

The vertical shift of the nomogram is mainly established during the period at which the moisture content is only slowly decreasing. In this part of the experiment a high accuracy of the determined weights is of greater importance.

The graphical representation of the experimental results during the course of the experiment makes it possible to take at the earliest possible moment another step in the series of suction forces.

The use of an e-function nomogram as described, will give greater control over the procedure during the experiment, result in a better use of the available manpower, reduce the laboratory space needed and increase the speed with which the results are made available to research workers.

1. Introduction

For the determination of the relation between the moisture content of a soil sample and the suction force, *i.e.* the moisture-retention or pF-curve, a series of equilibrium-moisture contents are measured when a series of constant suction forces is applied. The usual laboratory measuring technique is to establish the loss of moisture from

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the sample by means of successive weighings and after setting up a constant suction force. The end of a determination, *viz.* the point at which equilibrium is reached between moisture content and suction, is usually indicated by the fact that the moisture content in the sample did not substantially change between the last two weighings.
To take this criterion means, however, that only use is made of the last result of a series of measurements with one suction force. This is an inefficient use of all the data then known. Moreover, the repeated weighings of the sample will mean an equal number of interferences in the experiment. The desired equilibrium between suction and moisture content is furthermore at large suction forces only reached after a relatively long period. This method of determining the pF-curve therefore requires a great amount of time, man-power and laboratory space. The present paper describes a method by which these difficulties can largely be overcome.

2. The outflow as function of the time

The method to be discussed is based on the physical relation between the outflow at a constant suction force and time and was first described in a mimeograph of our Institute (I.C.W., 1962). RIJTEMA (1959) derived a formula for this relation in which the outflow from the sample Q(t) and the time t passed since applying the suction, are present as variables.
In the derivation of this formula the following assumptions were made:

- the capillary conductivity K remains constant with small changes of the suction force, and
- the relation between the moisture content P and the suction Ψ is, with such small changes, a linear one according to

$$\Psi = A + BP \tag{1}$$

The infinite series giving the solution converges very rapidly with increasing t, and can, neglecting the higher order terms, be written as

$$Q(t) = Q_\infty - Q_\infty \frac{2h^2 l^2}{\alpha^2 [hl (hl + 1) + \alpha^2]} \exp \left(- \frac{K}{B} \alpha^2 \frac{t}{l^2}\right) \tag{2}$$

where

$Q(t)$ = outflow after time t (cm^3).
K = capillary conductivity (cm/day).
h = $\frac{K_p}{l_p K}$ in which K_p = hydraulic conductivity of the medium on which the sample rests (cm/day).
l_p = thickness of that medium (cm).
l = length of the sample (cm).
B = slope of the moisture-retention curve according to (1).
α = constant ($\alpha \operatorname{tg} \alpha = hl$).
Q_∞ = total outflow for $t \to \infty$ (cm^3).

Outflow data will therefore approximately follow an equation of the exponential type provided t is taken not too small.
When using cylinders of 100 cc, the outflow Q(t) equals the change in the total weight of the sample and also the change in moisture content expressed as a percentage.
Take the weight of the dry matter of the sample to be G grams and the weight of

the sample-ring including accessories g grams. Let the volume of water remaining in the soil at time t be M_t. Then the total weight of the sample can be represented by

$$T_t = G + g + M_t \qquad (3a)$$

The initial total weight has an index t = i ; the weight at the asymptotical equilibrium state is given by T_∞. The moisture content in the sample decreases in time t with an amount of $(M_i - M_t)$ grams which is the same as $(T_i - T_t)$. The decrease in total weight equals in absolute value the increase of the outflow Q(t), or expressed algebraically :

$$T_i - T_t = - Q(t) \; , \; i \leqslant t_\infty \qquad (3b)$$

Using this relation, (2) can be applied to the weighing results and made suitable for use in routine determinations.

3. Conversion of the given formula

We are in this article mainly concerned with deriving a simple relation between moisture content and suction force, and therefore with the value of the equilibrium-moisture contents at each suction applied.
Converting the constants of (2) into one single symbol and using (3b), the following simple form for the relation between T_t and t is obtained

$$T_t - T_\infty = ae^{-\beta t} \qquad (4)$$

where

$$\beta = \frac{K}{B} \frac{\alpha^2}{l^2} \qquad (5)$$

The constant β is a measure for the capillary conductivity or the velocity with which a sample loses its moisture. A high value of β agrees with a high velocity.

4. Properties of the simplified formula

When t approaches infinity,

$$T_t = T_\infty$$

The asymptotical value T_∞ of T_t is therefore the total weight of the soil sample when equilibrium is reached between moisture content and suction force.
Moreover, with log on base e

$$\frac{\log a}{\beta} = a'$$

(4) can be written

$$T_t - T_\infty = \exp \left\{ -\beta \; (t - a') \right\} \qquad (6)$$

This last expression shows that when T_t is plotted on the ordinate and t on the abscissa of a co-ordinate system, translations parallel to the corresponding axis of T_t over a distance T_∞ and of t over a distance a′, will give the elementary form

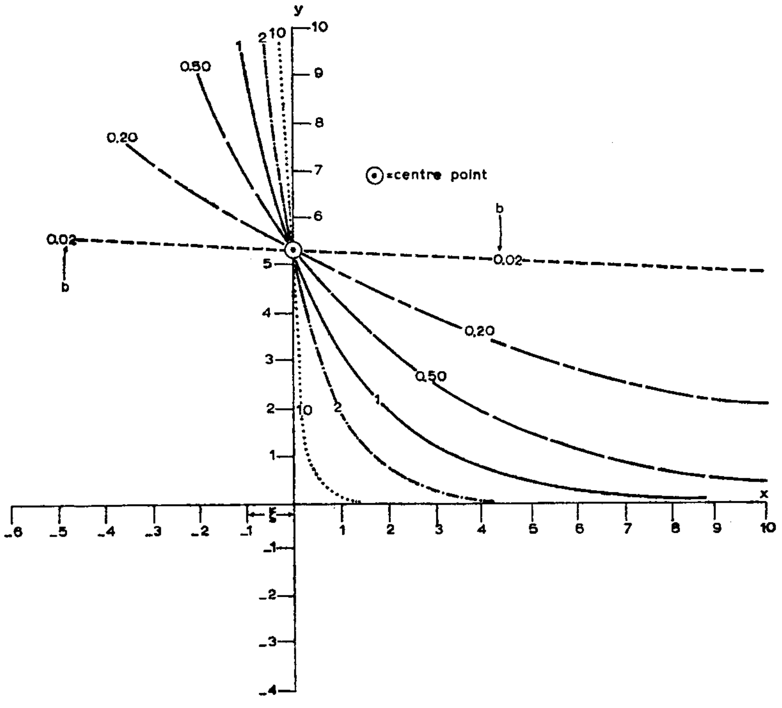
$$T_t' = e^{-\beta t'} \qquad (7)$$

the primes denoting that the above-mentioned translations have taken place.

In (7) only one parameter is present, *viz.* the one determining the curvature of the line. The expression can be represented by a family of curves of which every member is indicated by one specific value of the parameter β. If we take t' towards $+\infty$, then T_t' approaches zero, independent of the value of the parameter β. All curves of the family will therefore have the same asymptote as lower limit. Next, when $t' = 0$ the relation (7) will yield $T_t' = 1$, also independent of β, so all curves intersect for $t' = 0$ in the same point, the so-called centre point $(t', T_t') = (0,1)$.
Such a family of curves is shown in FIG. 1. The general form of these curves can be written as

$$y = (2e)e^{-\frac{1}{2}bx} \tag{8}$$

FIG. 1. Part of an e-function nomogram, based on the relation $Y = (2e) \exp(-0{,}5\ bx)$. The value of b only holds when $\xi = 1$ cm



The constants have been chosen in such a way that a convenient diagram can be obtained. For this reason the centre point has been shifted to (x, y) = (0, 2e).
The relation between the constants of (8) and (4), (6) and (7) is given by

$$b = 2\beta$$
$$y = T_t - T_\infty = T_t'$$
$$x = t - \frac{1}{\beta}\log\frac{a}{2e}$$
$$= t - a' + \frac{1}{\beta}\log 2e$$

$$= t' + \frac{1}{\beta} \log 2e$$

From $b = 2\beta$ and (5) follows (with D = Diffusivity)

$$D = \frac{bl^2}{2\alpha^2} \tag{9}$$

5. The use of the nomogram

If a graphical representation of an exponential function is available, the family of curves of the e-function nomogram, plotted on a transparent sheet, can be placed over the diagram of that function. Now move the centre point of the family of curves along the curve of the function until one of the curves of the family covers the curve in question, taking care to keep the co-ordinates of both systems parallel to each other.

A somewhat different method can be used when one has a series of experimental data to which a function of type (7) may be expected to apply. The data are plotted in a diagram. The centre point of the nomogram is then shifted along a fictitious line estimated to give the relation as indicated by the data. The curve to be selected from the family is the one that gives the best fit. When working in this manner it is, for routine research, advisable to judge visually whether the algebraic sum of deviations of the data, measured perpendicularly to the curve (8), is zero (see next section).

It is now possible to determine the values of the parameters T_∞ and a', which give the translation of the axes according to (6) and the curvature represented by b.

The value of T_∞ gives the total weight in the equilibrium state. The value of a' can be physically interpreted as the moment after which the soil sample has to lose 2e volume per cent of its moisture content to reach an equilibrium situation.

6. Propagation of errors

The curve which best fits the data, is given by equation (6). The implicit form can be written

$$F = T_t - T_\infty - \exp\{-\beta (t - a')\} \tag{10}$$

Now $F = 0$ if the observation (t, T_t) lies on the fitted curve. Data above the curve yield $F > 0$, data beneath the curve $F < 0$. The effect of small errors $\triangle t$ and $\triangle T$ on F is given by:

$$\triangle F = \triangle T + \beta \exp\{-\beta (t - a')\} \triangle t \tag{11}$$

using TAYLOR's series (DEMING, 1948). In (11) the parameters have constant finite values.

The influence of small errors in t and T_t can be ascertained by means of (11). For large values of t the effect on F is approximately

$$\triangle F = \triangle T$$

In these situations accurate weighings must therefore be made. This relation holds when t is not too small and β is large, *i.e.* at a high rate of moisture loss.

When (t — a') is small, the error in F is given by

$$\triangle F = \triangle T + \beta \triangle t$$

which, for large values of β, can be approximated by

$$\Delta F = \beta \Delta t$$

Under these conditions, *viz.* samples with a large capillary conductivity, the time at which the first weighings are made must be recorded very accurately.
The geometric interpretation of these considerations can be readily found in FIGS. 2—4.

7. Application of the nomogram to moisture-content determinations

The measuring starts at the moment when a certain suction force is applied to the sample. The results are taken down by plotting in a diagram the total sample weight T (ordinate) and the time t that has elapsed since starting constant suction (abscissa) (see FIGS. 2—4).
Calculations to follow subsequent measurements do not have to be made and it is not necessary to wait until the asymptotical value T_∞ has been reached, procedures that are a disadvantage of other methods, as for example that of KUNZE and KIRKHAM (1962).
After having carried out several determinations with one constant suction, the correspondence with the physical relationship (2) can be found by means of the nomogram. The curve with the best fit can be copied or interpolated from the nomogram; the asymptote T_∞ will indicate at which value the equilibrium situation can be expected to occur.
To determine the three constants of (6), *viz.* T_∞, a′ and $\beta = \frac{1}{2}b$, three observations would be sufficient. Generally one will prefer, however, to have a somewhat larger number of data to fit the curve.
To determine the capillary conductivity in accordance with RIJTEMA (1959), the results of the fitted curve can now be used. Let T_o be the intercept of the fitting curve with the y-axis, then

$$\frac{Q_\infty - Q_o}{Q_\infty} = \frac{T_o - T_\infty}{T_i - T_\infty}$$

A table of values for α^2 and hl corresponding to this ratio are given in I.C.W. (1962). With V as the volume of the sample and $\Delta\,\Psi$ as the increment of the suction, the slope B of the moisture-retention curve is given by

$$B = \frac{T_i - T_\infty}{V \Delta \Psi}$$

With (9), the capillary conductivity can be determined from

$$K = DB$$

When the measurements are completed, the dry weight G and the weight of the sample ring g is determined. The diagram can be completed by constructing a scale in volume per cent according to (3a).

Some examples

The FIGS. 2—4 give examples of the described method. The behaviour of several samples can be readily compared without the need of quantitatively determining the constants of (2).

The example given in FIG. 2 relates to a sandy clay soil. After saturation and free leaking of the sample, the total weight was 282.5 g which after calculations at the end of the measurements was found to correspond to a moisture content of 36 % at a pF-value of 0.4.

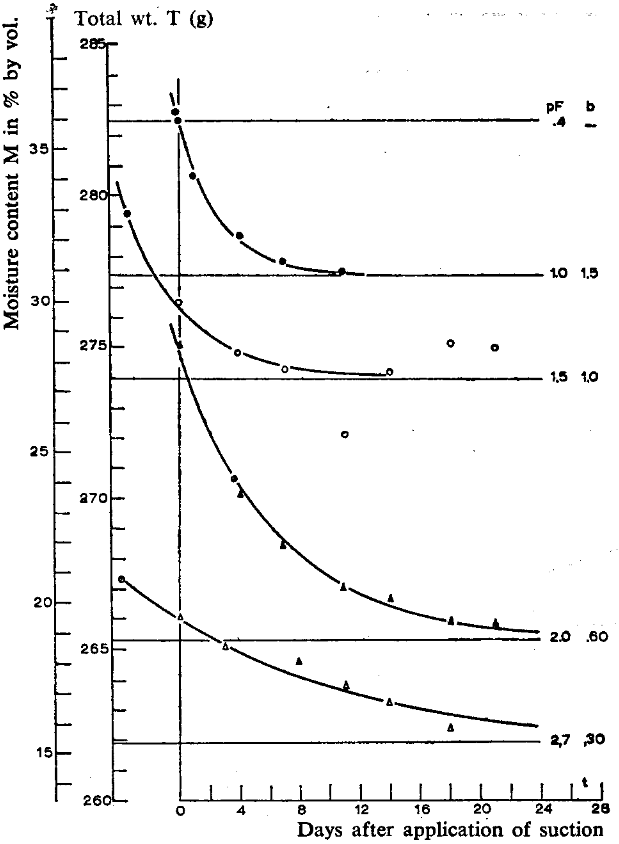


FIG. 2
Moisture loss of a sandy clay soil at subsequent constant suction forces (pF 0.4; 1.0; 1.5; 2.0; 2.7). Circled dot = centre point of nomogram (FIG. 1)

A suction corresponding to pF 1.0 was then set up. The way in which the weight of the sample decreased is shown in the upper part of the diagram. The determinations with this suction force ended when a moisture content of 30.8 % was reached. Now a pF-value of 1.5 was taken, the results being given in the second curve from the top in FIG. 2. A suction corresponding to a pF-value of 2.0 and 2.7 was then applied. For the latter value the moisture content in the equilibrium state was 15.4 % by volume.

Inspection of the entire diagram shows that the data are arranged rather well along the curves of the functional relation (2), except for some points at pF = 1.5.

FIG. 3 illustrates a similar series of curves relating to a sample from a field of reworked clayey marine sand. The rapid loss of moisture at pF = 1.0 is striking. The value 10.0 of the parameter b indicates a large capillary conductivity. An irregularity is shown in the range for pF = 2.0. It is probable that during these measurements

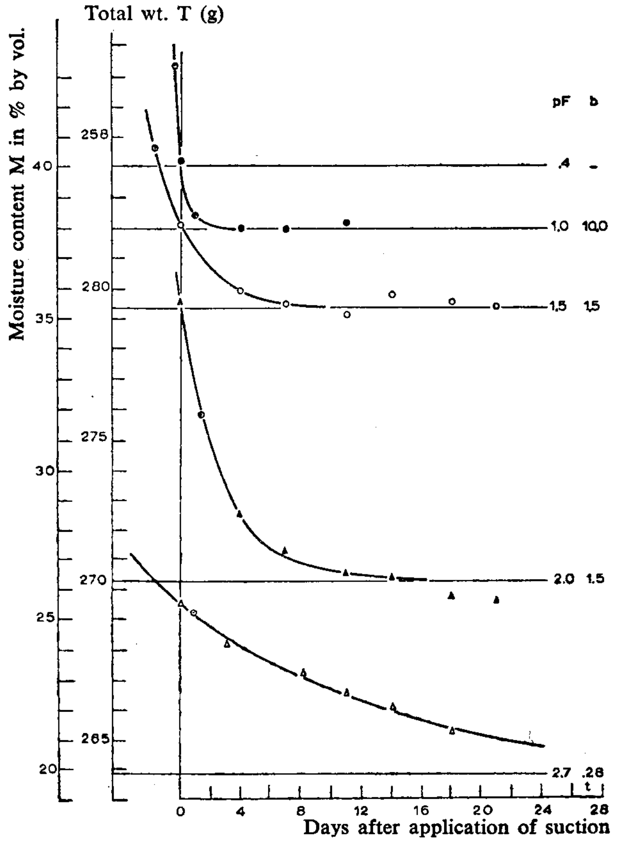


FIG. 3
Moisture loss of a re-worked clayey marine sand

evaporation of soil moisture occurred. In this sample, the capillary conductivity decreases at higher suction forces and the sample loses its moisture more slowly.

FIG. 4 shows curves pertaining to a sticky clay soil.

The examples selected can be easily compared in a more absolute way by shifting the moisture-content scales to the same level. It can then immediately be seen that the sticky clay has a higher moisture content, approximately 53 % at pF = 0.4, but the levels of the asymptote show that a small amount of moisture is lost at higher pF-values. At pF = 2.7, the moisture content is still more than 45 %, but the rate at which the moisture is lost is fairly high.

The other two soils have a lower moisture content, although more of it becomes available with an equal increase in the suction. The rate at which this occurs is in general lower than for the sticky clay, with the exception of the pF-value 1.0 applied to the marine sand. There the asymptotical value is almost reached within two days.

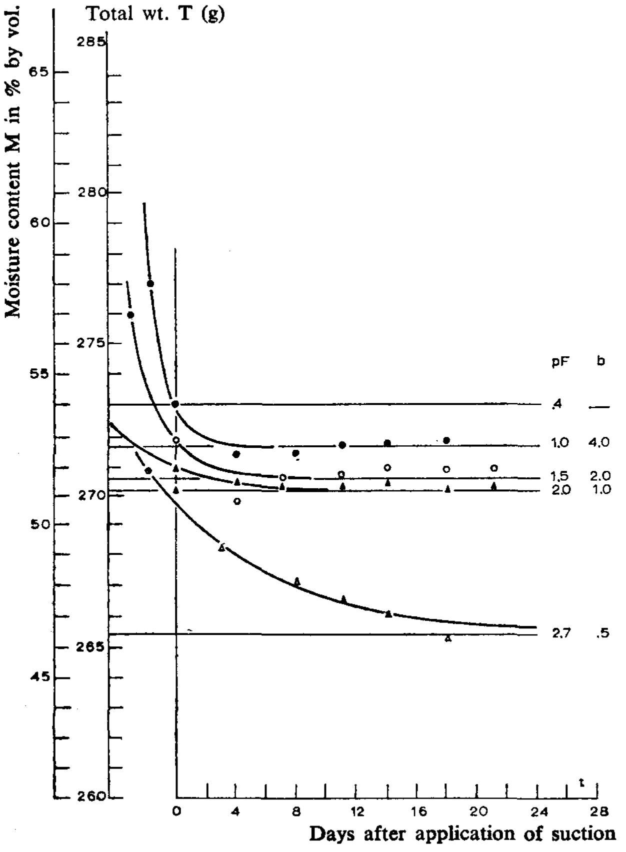


FIG. 4
Moisture loss of a sticky clay soil

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