ON THE EVALUATION OF THE THERMAL (OR EDDY) DIFFUSIVITY FROM THE DIURNAL TEMPERATURE WAVE

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SUMMARY

The theory of the evaluation of the thermal (or eddy) diffusivity, a, from determinations of the amplitude and phase of the diurnal temperature wave in a non-homogeneous medium is briefly discussed. It is shown by an example pertaining to the lower air layers that the often used procedure of applying the theory for a homogeneous medium to successive layers of a non-homogeneous medium may lead to values of a that are in error by more than a factor ten.

THEORY

The evaluation of the thermal diffusivity, a, of soil or air from measurements of the diurnal or annual temperature waves is one of the classical examples of the application of the mathematical theory of heat conduction to a problem occuring in soil physics and in micrometeorology. It is based on the application of the FOURIER-equation in one dimension for an isotropic medium:

$$\frac{\delta}{\delta z}\left(\lambda \frac{\delta T}{\delta z}\right) = C \frac{\delta T}{\delta t}. \qquad (1)$$

where T = temperature, t = time, z = vertical coordinate, λ = thermal conductivity, C = volumetric heat capacity. The solution of this equation is sought for a harmonic variation of temperature with time, i.e.:

$$T\,(z,t) = T_a + A(z)\ \exp.\ (i\omega t + \varphi\,(z)), \qquad (2)$$

where T_a is an average temperature, which will be considered as independent of z and t. Further the amplitude is assumed to be zero at $z = \infty$. If both λ and C are independent of z and t the well-known solution to this problem is given by:

$$T\,(z,t) = T_a + A(\text{o})\ \exp\left\{ i\omega t - (1 + i)\ z/(2a/\omega)^{\frac{1}{2}} \right\}, \qquad (3)$$

where $a = \lambda/C$ and $\varphi(\text{o})$ has arbitrarily been taken as zero.

With applications to soil and air λ and C are usually functions of z and t, however. The difficulty of the variation with height is usually evaded by applying the solution Eq. (3) to successive layers. In doing so it is presumably expected that the errors introduced by the non-constancy of λ and C will be small, provided that comparatively thin layers are considered.

This procedure was criticized as early as 1919 by AICHI in connection with applications to soils and more recently by COWLING and WHITE (1941) with reference to data pertaining to the lower air layers. The correct way to treat the problem of a soil consisting of two homogeneous layers was given by

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PEERLKAMP (1944).

It was shown by AICHI (1919) and independently by PEERLKAMP (1944) that if A and φ are known functions of z, λ and C can be found from the equations:

$$\frac{\lambda'}{\lambda} = \frac{f^2 - g^2 - f'}{f} \cdot \frac{\lambda}{C} = \frac{f\omega}{g\,(f^2 + g^2) + f'g - fg'}, \qquad (4)$$

where a prime denotes a differentiation with respect to z, $f = -A'/A$ and $g = -\varphi'$. Unfortunately the practical value of these equations is small, as they contain second derivatives of A and φ', which cannot be determined with a sufficient degree of accuracy, unless very detailed measurements of the profile are made.

If on the other hand λ and C are known functions of z of certain types analytical solutions of Eq. (1) with the appropriate boundary conditions can be found. An example of this, which pertains to the lower air layers, is the case where C is independent of height, while the diffusivity can be represented by a function of the type:

$$a = \alpha\,(z + z_o)^{\beta}. \qquad (5)$$

The solution in question is now a linear combination of the two functions:

$$(z + z_o)^{\frac{1-\beta}{2}}\; Z_{\pm\frac{1-\beta}{2-\beta}} \left\{ \frac{2\,(z + z_o)^{\frac{2-\beta}{2}}}{2-\beta} \left(\frac{-i\omega}{\alpha}\right)^{\frac{1}{2}} \right\} \exp\,(i\omega t),$$

where Z_p denotes a BESSEL function of order p. A numerical example is discussed in the following section.

EXAMPLE AND DISCUSSION

The example is based on experimental data of BEST (1935), who measured airtemperatures at Porton at heights of 2.5, 30, 120, 710 and 1710 cm. A harmonic analysis of the diurnal variation of the average temperature in June led to the values of the amplitude and phase for the first harmonic given in Table 1. From these data BEST deduced the eddy diffusivity by application of Eq. (3) to successive layers. The results are represented in Table 2; a_A denotes the value of the diffusivity derived from the ratio of amplitudes, while a_φ follows from the phase differences.

Now two different variations of a with height were chosen in such a manner that an analytical solution could be found, while the resulting values of A and φ resembled those found by BEST. In the first case a in cm²/sec is represented by the equations:

$$a = 7\,(z + 2), \text{ for } z = \leqslant 710; \quad a = 4984, \text{ for } z = \geqslant 710. \qquad (5)$$

An exact solution which has the same amplitude and phase at $z = 710$ as found by BEST is:

$$T = T_a + 1.18\; H_o^{(2)} \left\{ 6.45 \times 10^{-3}\; (-i)^{\frac{1}{2}}\; (z + 2)^{\frac{1}{2}} \right\} \exp\,(i\omega t - 1.031\,i) +$$

$$+ 2.83\; J_o \left\{ 6.45 \times 10^{-3}\; (-i)^{\frac{1}{2}}\; (z + 2)^{\frac{1}{2}} \right\} \exp\,(i\omega t - 0.561\,i), \text{ for } z = \leqslant 710. \qquad (6a)$$

$$T = T_a + 4.10\, \exp \left\{ i\omega t - 0.310\,i - (1 + i)\,(z - 710) / 1.171 \times 10^4 \right\}, \text{ for } z = \geqslant 710. \qquad (6b)$$

Here T is in °C. $H^{(2)}$ is the second HANKEL function and J the BESSEL function of the first kind. The values of A and φ computed from this solution are given in Table 1.

The second case is as follows:

$$a = 0.78\,(z+2)^{\frac{4}{3}},\ \text{for}\ z \leqslant 710;\qquad a = 4960,\ \text{for}\ z \geqslant 710. \tag{7}$$

The solution which gives the same temperature wave at $z = 710$ is:

$$T = T_a + 62.2\,(z+2)^{-\frac{1}{3}}\ \exp\left\{i\omega t + 1.707\,\mathrm{i} - 0.0205\ (1+i)\ (z+2)^{\frac{1}{3}}\right\} +$$

$$+\,59.0\,(z+2)^{-\frac{1}{3}}\ \exp\left\{i\omega t - 1.278\,i + 0.0205\ (1+i)\ (z+2)^{\frac{1}{3}}\right\},\ \text{for}\ z \leqslant 710, \tag{8a}$$

$$T = T_a + 4.10\ \exp\left\{i\omega t - 0.310\,i - (1+i)\ (z-714)\,/\,1.168\times 10^4\right\},\ \text{for}\ z \geqslant 710. \tag{8b}$$

The values of A and φ for this case are also represented in Table 1.

Table 1. Values of the amplitudes, A (in °C), and the phase φ (in radians) of the temperature waves at various heights.

Height (cm)	Best		Case 1		Case 2	
	A	$-\varphi$	A	$-\varphi$	A	$-\varphi$
2.5	5.78	0.000	5.55	0.055	8.26	−0.155
30	5.14	0.129	4.95	0.138	5.64	0.046
120	4.72	0.188	4.57	0.207	4.75	0.176
710	4.10	0.310	4.10	0.310	4.10	0.310
1710	3.76	0.396	3.77	0.396	3.77	0.396

Table 2. Values of the thermal diffusivity (in cm²/sec) derived from Eq. (3) (a_A from amplitude ratios, a_φ from phase differences [1])), compared with the values given by Eqs. (5) and (7).

Height (cm)	Best		Case 1			Case 2		
	a_A	a_φ	a_A	a_φ	a	a_A	a_φ	a
2.5	2.1	1.7	2.1	4.0	31.5	0.19	0.68	5.8
30	40	83	46	62	224	10.0	17.4	79
120	650	850	1074	1194	854	584	705	472
710	4900	5000	4984	4984	4984	4960	4960	4960
1710					4984			4960

[1]) These values refer to the air layers between the heights mentioned in the first column.

The values of a derived from Eq. (3) are listed in Table 2 together with the "actual" values, following from the Eqs. (5) and (7). It can be seen that the former are greatly in error, below 30 cm by even more than a factor of ten. The error may also be large in spite of the fact that the difference between a_A and a_φ is comparatively small as e.g. for the layer of 120 to 710 cm in case 1. The physical reason for the discrepancies lies in the fact that the condition of continuity of the heat flux is not satisfied if Eq. (3) is applied tot successive layers. A fuller discussion of the problem will be published elsewhere.

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ON THE TIME DEPENDENCE OF THE EDDY HEAT CONDUCTIVITY IN THE LOWER AIR LAYERS FROM THE RATIO OF THE AMPLITUDES OF THE DIURNAL AND ANNUAL TEMPERATURE WAVES AT 200 AND 10 CM

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SUMMARY

A relation of the type: $a = \alpha (z + z_0)^{\beta}$, in which a is the thermal diffusivity in air, z the height and z_0 a roughness parameter, has been used to calculate the ratio of the amplitudes of the temperature waves at 10 and 200 cm height for values of α ranging from 0.3 to 3×10^3 and values of β ranging from 0.5 to 1.4. It was found that this ratio is only slightly affected by variations in α and β, if $\beta < 0.9$.

Experimental results at Wageningen show that the value of β lies close to 1. No time dependence was found in course of a day or in course of a year.

INTRODUCTION

The FOURIER-equation for the conduction of heat in a semi-infinite, homogeneous medium, in which the temperature, T, is a function of height, z, and of time, t, is:

$$\frac{\delta}{\delta z}\left(\lambda \frac{\delta T}{\delta z}\right) = C \frac{\delta T}{\delta t} \tag{1}$$

The volumetric heat capacity of air, C, is considered as a constant, but the conductivity, λ, besides being dependent upon atmospheric stability, is a function of height. It is assumed that the dependence of λ on height can be represented by (cf. SUTTON (1953) p. 215):

$$\lambda = b (z + z_0)^{\beta} \tag{2}$$

The following boundary conditions must be satisfied:

1 at zero height T is a harmonic function of time with amplitude A_0 and angular frequency ω:

$$T(o,t) = T_a + A_0 e^{i\omega t} \tag{3a}$$