

## **A modified Mitscherlich equation for rainfed crop production in semi-arid areas: 1. Theory**

K. HARMSSEN

International Institute for Aerospace Survey and Earth Sciences (ITC), PO Box 6,  
NL-7500 AA Enschede, The Netherlands (fax: +31-53-4874200; e-mail: harmsen@itc.nl)

Received 25 November 1999; accepted 22 October 2000

### **Abstract**

The classical Mitscherlich equation is based on Liebig's Law of the Minimum and describes the yield response of a crop to an increase in the main factor that is limiting growth. The maximum, or potential, yield is an important parameter in the Mitscherlich equation and is assumed to be constant, that is, not affected by other factors that limit actual yields under field conditions. The assumption that potential yield is a constant does not apply to rainfed agriculture in semi-arid regions because under such conditions potential yields vary with crop-available moisture. A theoretical framework for the application of the Mitscherlich equation to rainfed crop production is presented. Water-limited potential yield is assumed to be a linearly increasing function of available moisture. Similarly, the quantity of nutrients required by a crop to achieve water-limited potential yield is assumed to be a linearly increasing function of seasonal rainfall. Finally, nutrient availability is also thought to depend on available moisture. The general form of the modified Mitscherlich equation for response to nutrients is simplified, by expressing all moisture dependent parameters as functions of annual rainfall.

*Keywords:* nutrient availability, nutrient-use efficiency, nutrient uptake, potential yield, water balance

### **Introduction**

ετος φερει, ουτι αρουρα  
(*The harvest is the year's and not the field's*)  
Ancient Greek proverb (Theophrastus, 1989, 1990)

The classical Mitscherlich equation is based on Liebig's Law of the Minimum and describes the response of a crop to an increase in the factor that is limiting growth, other factors being constant (Von Liebig, 1855; Mitscherlich, 1909, 1913). Although Mitscherlich's equation was assumed to apply to all factors that could limit growth ('vegetation-factors', 'growth-factors' or 'production factors'), such as light, temperature, water and nutrients, it is most commonly applied to nutrients, in particular, nutrients supplied in the form of chemical fertilizers or organic soil amendments.

The assumption that the potential, or maximum, yield of a particular crop is a constant across sites and seasons may be approximately valid for a country with a humid temperate climate, with limited relief, where moisture is generally not limiting growth and radiant energy and temperature do not vary much between sites and seasons. However, for rainfed agriculture in semi-arid regions of the world, this assumption would not hold. Under such conditions, potential yields of rainfed crops are essentially water-limited: they range from practically zero, under very dry conditions, to values similar to those achieved under irrigation, under high rainfall conditions.

The Mediterranean environment is a typical example of a semi-arid environment where actual yields in farmers' fields vary with seasonal rainfall. John L. Monteith stated that 'in many types of climate, fluctuations of rainfall and temperature are the main sources of variability in yield but it was presumably the particular experience of Mediterranean farmers which was summed up long ago in one simple statement: *annus fructum fert, non tellus* (Monteith, 1981). The statement referred to by Monteith is derived from Theophrastus (372–287 BC), the successor of Aristotle in ancient Greece, who cites an ancient Greek proverb stating that 'the harvest is the year's and not the field's' (Theophrastus, 1989, 1990). Incidentally, it may be noted that Theophrastus is quite well aware of the effect of soil fertility and agronomic management on crop yields, but seems to support the view that ultimately available moisture determines (water-limited) potential yield.

Production functions, such as the Mitscherlich equation, are widely used in agricultural science to relate yield response to nutrient supply and other factors (De Wit, 1992, 1994). As time is not a variable in such production functions, the dynamics of, for example, nutrients and water, and their interaction with crop physiology during the growing season, cannot be considered. Therefore, if the interaction between crop development and available moisture is very complex or significantly affects final crop yield, a production function may not be adequate and one may have to resort to deterministic or stochastic simulation modelling or a comparable approach.

Interest in the Mitscherlich production function was stimulated by C.T. De Wit, who observed that the 'law of diminishing returns' could not explain the near linear (or even increasing) response of crop yield to increased use of resources, in particular, chemical fertilizer, in the industrialized world after 1945 (De Wit, 1992, 1994). The observation of De Wit could be explained by assuming that any 'diminishing returns' with regard to a particular resource, such a nitrogen fertilizer, would be compensated by increased efficiency as a consequence of other technical changes in the production process (De Wit, 1992, 1994). De Wit's paradigm of 'compensation of diminishing returns' turned out to be similar to Liebscher's Law of the Optimum (Liebscher, 1895), which presumably was introduced in the Dutch agronomic literature by F. Van Der Paauw (1938) and O. De Vries (1939). The Mitscherlich equation may be considered a special case of Liebscher's Law of the Optimum (De Wit, 1992) and, in fact, as its mathematical formulation, in particular in the form of the multifactorial Mitscherlich-Baule equation (Baule, 1917; Mitscherlich, 1956).

The Mitscherlich-Baule equation has been further developed and modified (e.g., Von Boguslawski & Schneider, 1962, 1963). These modifications have broadened the range of applicability of Mitscherlich-Baule-type production functions, although

the physical significance of some of the parameters introduced may need further study. In the present paper, the discussion will be limited to the original Mitscherlich-Baule equation, as the emphasis is on the introduction of water-limited potential yield and nutrient-moisture interactions in the production function and this does not require more complex forms of the production function.

The aim of this paper is to develop a form of the Mitscherlich-Baule equation which considers explicitly the effect of available moisture on potential yield, nutrient demand and on nutrient availability. Such an equation would be relevant to rainfed crop production in semi-arid regions. In a companion paper (Harmsen, 2000), the moisture dependence of potential yield and of nutrient availability will be discussed for nitrogen and phosphorus, using data from agronomic experiments in the semi-arid region of Syria.

### Theoretical

Yield response to application of a growth-limiting nutrient can be described by a Mitscherlich equation of the form:

$$\partial Y / \partial N_f = \varepsilon_f (Y_x - Y) \quad (1)$$

where  $Y$  is the total biological or total dry matter (dm) yield of a particular crop ( $\text{kg dm ha}^{-1}$ ), which is a function of more than one variable,  $Y_x$  is the maximum, or potential, yield of that crop under the climatic and edaphic conditions of the experiment,  $N_f$  is the rate of the nutrient applied to the crop ( $\text{kg nutrient ha}^{-1}$ ) and  $\varepsilon_f$  is an 'activity' coefficient ( $\text{kg}^{-1} \text{ nutrient ha}$ ), which is a measure of the availability of the applied nutrient to the crop.

The differential equation (1) represents the slope of the production function and is, in fact, the definition of 'nutrient-use efficiency' ( $\text{kg dm kg}^{-1} \text{ nutrient}$ ). Mitscherlich thus assumes that the nutrient-use efficiency is proportional to the difference between potential and actual yield. If actual yield is very low, the response to application of the nutrient, which is limiting growth, is highest. When the actual yield approaches the potential yield, the response to the limiting nutrient tends to zero.

Equation (1) shows that the nutrient-use efficiency, that is, the slope of the yield response curve, depends on the value of  $Y_x$ : in the limiting case for  $Y = 0$ , the slope equals  $\varepsilon_f Y_x$ , that is, increases linearly with increasing values of  $Y_x$ .

The notion 'potential yield' refers to the maximum yield that can be achieved under clearly specified agro-ecological, edaphic and management conditions. Strictly speaking, 'potential yield' would refer to a situation where only the genetic potential of the crop would limit yield. In practice, however, the term potential yield is also used in situations where photoperiod, temperature, carbon dioxide or radiant energy are limiting yield. Such use of the term potential yield requires that the environmental conditions to which the term 'potential' refers, should be carefully specified. For example, the 'potential yield' of a crop grown under rainfed conditions at a certain location during the rainy season may be lower than the potential yield of the same

crop grown at the same location during the post-rainy season (e.g., using stored soil moisture or under irrigation), when radiant energy levels and temperatures may be more favourable.

In the present paper, the term 'water-limited potential yield' will be used for crops grown under rainfed conditions, that is, conditions where ultimately available water determines potential yield. Of course, these water-limited potential yields are generally not achieved under farmers' field conditions, as other yield-limiting factors, such as available nutrients, yield-reducing factors, such as weeds, pests and diseases, or agronomic factors, such as seedbed preparation, seeding rate, plant density or row spacing, may determine actual yields in farmers' fields.

Equation (1) gives upon integration:

$$Y = Y_x - (Y_x - Y_o)\exp(-\epsilon_f N_f) \quad (2)$$

where 'exp' is the exponential function and  $Y_o$  is the yield when no nutrient is applied in the form of an external source ( $N_f = 0$ ). If the nutrient availability from all internal (soil) and external sources is zero, then  $Y_o$  is likely to be zero. The Mitscherlich equation thus assumes that if the nutrient under consideration is the factor most limiting crop yield and if other factors affecting crop growth are held constant at near-optimum levels, yield increases with increasing nutrient rate, until it asymptotically reaches a maximum value.

Assuming that potential yield is a function of available moisture, Equation (2) may be written as:

$$(Y_o - Y)/Y_o = \exp(-\epsilon_t N_t) \quad (3)$$

where  $Y_o$  denotes the potential yield as a function of available moisture ( $\theta$ ),  $N_t$  the total quantity of nutrients available to the crop from all sources,  $\epsilon_t$  an activity coefficient which is a measure of the nutrient availability associated with  $N_t$  and where it has been assumed that  $Y_o = 0$  at  $N_t = 0$ .

Equation (3) would imply that a constant amount of available nutrients ( $N_t$ ) would give different yield responses, depending on  $Y_o$ , that is, on available moisture conditions. Or, differently stated, that the absolute amount of nutrient needed to reach a certain fraction of the maximum yield is the same whether yields are high or low (De Wit, 1992). De Wit refers to this feature of the Mitscherlich model as the 'constant activity' of the production factor  $N_t$  (De Wit, 1992, 1994). Although De Wit is of the opinion that nutrient-use efficiency may increase with increasing potential yield, he refers to the assumption of constant activity as a 'heroic' assumption, thus questioning its general validity.

It is conceivable that nutrient-use efficiency (cf. Equation 1) increases with increasing water-limited potential yield. This is also implied in Equation (1), if  $Y_x$  is replaced by  $Y_o$ . However, it can easily be shown that Equation (3) could result in the hypothetical situation that at high potential yield levels, more nutrients would be taken up by the crop than would be available in the form of soil nutrients or applied in the form of fertilizer or some other external source. In principle, it is conceivable

that at higher rainfall levels, more soil nutrients are available to the crop, such that total nutrient uptake could increase, but nutrient uptake beyond the level of total available nutrients is, of course, highly unlikely.

De Wit refers to the Mitscherlich equation ('constant activity') as a special case of Liebscher's 'Law of the Optimum', which states that 'a production factor that is in minimum supply contributes more to production the closer other production factors are to their optimum' (Liebscher, 1895; De Wit, 1992, 1994). In De Wit's view, the difference between Liebig's Law of the Minimum and Liebscher's Law of the Optimum, for the yield response ( $Y$ ) to a single production factor ( $N$ ), would be in the slope of the production function ( $\partial Y/\partial N$ ) as related to the potential yield ( $Y_0$ ): In the case of Liebscher the slope of the production function, in the limit for  $N=0$ , would be a linear function of  $Y_0$  (e.g.,  $\epsilon_i Y_0$ ), whereas in the case of Liebig the slope would be the same for all values of  $Y_0$ , in the limit for  $N=0$ . The Mitscherlich equation (1) can be modified to fit the Liebig model:

$$\partial Y/\partial N_i = \epsilon_{iL}(1 - Y/Y_0) \quad (4)$$

which gives upon integration:

$$Y/Y_0 = 1 - \exp(-\epsilon_{iL} N_i/Y_0) \quad (5)$$

where  $Y_x$  has been replaced by  $Y_0$  and  $\epsilon_{iL}$  (kg dm kg<sup>-1</sup> nutrient) denotes the activity coefficient related the total-nutrient content ( $N_i$ ) for the Liebig model and where it is further assumed that  $Y_0 = 0$  if  $N_i = 0$ . From Equation (4) it follows that the slope of the production function (i.e., nutrient-use efficiency) equals  $\epsilon_{iL}$  in the limit for  $N_i = 0$ , for all values of  $Y_0$ . From Equation (5) it follows that, in contrast to the 'heroic' assumption of 'constant activity', the yield response to the application of a constant amount of nutrients depends on the value of  $Y_0$ , i.e., decreases with increasing  $Y_0$ .

Otto De Vries (1939) reviewed a large number of fertilizer response trials conducted in the Netherlands and concluded that some results could be better described by a Liebig-type model (Equation 5), whereas others were closer to a Liebscher-Mitscherlich-type model (Equation 3). The trial results did not clearly support either model, although the results of pot experiments tended to be closer to a Liebig-type model and field experiments were better described by Liebscher-Mitscherlich-type production functions (De Vries, 1939).

Although the original Mitscherlich equation was assumed to apply to a single production factor (Mitscherlich, 1909, 1913), it was soon realized that a similar approach could be followed with regard to more production factors (Baule, 1917, Mitscherlich, 1956):

$$Y/Y_x = (1 - \exp(-\epsilon_1 N_1))(1 - \exp(-\epsilon_2 N_2)) \dots (1 - \exp(-\epsilon_n N_n)) \quad (6)$$

where  $\epsilon_1$ ,  $\epsilon_2$ , and  $\epsilon_n$  are activity coefficients associated with  $N_1$ ,  $N_2$  and  $N_n$ , and where  $Y_0 = 0$  at  $N_i = 0$ . Limiting the discussion to 2 production factors, water ( $\theta$ ) and a nutrient ( $N_i$ ), it follows that:

$$Y/Y_x = (1 - \exp(-\epsilon_\theta \theta))(1 - \exp(-\epsilon_t N_t)) \quad (7)$$

Although it would be attractive to include the effect of water (c.q. annual rainfall) in the form of an exponential factor, as is done in the Mitscherlich-Baule equation (7), it is unlikely that  $\epsilon_t N_t$  in Equation (3) would be independent of rainfall if  $Y_\theta$  is strongly dependent on rainfall. In fact, one would expect the quantity of nutrients required to achieve water-limited potential yield to increase with increasing rainfall (cf. Equation 5). Also, the nutrient availability would be expected to be affected by rainfall conditions, albeit possibly in different ways for different nutrients, e.g., increase with rainfall in the case of phosphorus, because of increased mobility of phosphorus in the soil, but decrease with rainfall in the case of nitrogen, because of increased leaching and denitrification of nitrate.

Hence, under conditions of rainfed crop production, the exponential function would have to be modified to account for the effect of available moisture. This could be done as follows:

$$Y = Y_\theta - Y_\theta \exp(-\epsilon_{t,\theta} N_t / Y_\theta) \quad (8)$$

or, in differential form:

$$\partial Y / \partial N_t = \epsilon_{t,\theta} (1 - Y / Y_\theta) \quad (9)$$

where  $\epsilon_{t,\theta}$  is the moisture-dependent activity coefficient associated with  $N_t$  and where  $Y_\theta$  has been introduced in the exponential function (cf. Equations 4–5). Hence, it is assumed, in a way similar to the Liebig model, that the parameter  $N_t / Y_\theta$  is driving the nutrient uptake under water-limited growth conditions, rather than  $N_t$ . In other words, a constant amount of nutrient applied would have a relatively larger effect at low rainfall conditions, where the crop nutrient requirement would be relatively low, than at higher rainfall conditions, where the crop's requirement would be higher. Finally, the effect of soil moisture on the nutrient-availability in the soil and, thus, on the nutrient-use efficiency, is introduced through  $\epsilon_{t,\theta}$ . Moisture influences mineralization-immobilization, adsorption-desorption and dissolution-precipitation reactions in soil, nutrient transport by mass flow and diffusion to the plant roots, as well as root development and root activity itself. Also, the volume of the soil accessible to the roots of the crop increases. Therefore it seems plausible that the activity coefficient  $\epsilon_{t,\theta}$  depends on soil moisture conditions.

In the form of Equation (8), the modified Mitscherlich equation is difficult to evaluate, as  $Y_\theta$  and  $\epsilon_{t,\theta}$  are as yet unknown functions of  $\theta$ . In the following sections, approximate expressions for  $Y_\theta$  and  $\epsilon_{t,\theta}$  will be presented, such that the resulting production function can be tested in field experiments.

#### *Relation between water-limited potential yield and rainfall*

From Equation (7) it follows that the water-limited potential yield can be represented by:

$$Y_{\theta} = Y_x(1 - \exp(-\epsilon_{\theta}\theta)) \quad (10)$$

or, in a Taylor-series approximation, for small values of  $\theta$ :

$$Y_{\theta} = Y_x(\epsilon_{\theta}\theta - \epsilon_{\theta}^2\theta^2/2 + \dots) \quad (11)$$

that is, over a limited range of values of  $\theta$ ,  $Y_{\theta}$  can be approximated by a linear function of  $\theta$ . The simplest way to measure 'water' is by measuring annual rainfall,  $r$  (mm year<sup>-1</sup>). It should be noted that 'annual' rainfall may not be measured on the basis of a calendar year, but from the middle of the dry season to the middle of the next dry season, in the case of a uni-modal annual rainfall distribution. In practice, in many experiments in the semi-arid regions, only seasonal rainfall is measured, as most of the early or late rains do not contribute significantly to biomass production, since the scattered rainfall that reaches the soil during the dry season evaporates. In the present treatment, however, it will be assumed that 'annual rainfall' refers to a full 12-month period, centered around the rainy season.

Biomass production only occurs when annual rainfall exceeds a certain threshold value,  $r_0$  (mm year<sup>-1</sup>), below which no germination occurs or no biomass is produced, because rainfall is too low or too scattered throughout the season for crops to survive (Harmsen *et al.*, 1983; De Wit, 1994). Replacing  $\theta$  by annual rainfall, Equation (11) can be approximated by:

$$Y_{\theta} = Y_x\epsilon_{\theta r}\Delta r \quad (12)$$

where:

$$\Delta r = (r - r_0) \quad (13)$$

and where  $\epsilon_{\theta r}$  is an activity coefficient which is a measure for the availability of rainfall to the crop (mm<sup>-1</sup> year) and  $Y_x$  is potential yield (kg dm ha<sup>-1</sup> year<sup>-1</sup>). Of course, Equation (12) would only be valid for a limited range of rainfall levels, that is, yield may increase linearly with rainfall over a limited range, but at some level the relationship would break down because beyond that level rainfall would be no longer limiting yield.

The expression  $Y_x\epsilon_{\theta r}$  may be referred to as the crop water-use efficiency (kg dm ha<sup>-1</sup> mm<sup>-1</sup>). It may be noted that  $Y_x\epsilon_{\theta r}$  may also be defined on the basis of grain yield (kg grain ha<sup>-1</sup> mm<sup>-1</sup>) rather than total dry matter. Also, from an agronomic point of view, it may be desirable to define  $Y_x\epsilon_{\theta r}$  on the basis of crop evapotranspiration rather than rainfall (e.g., Cooper, 1983; Cooper *et al.*, 1983), but for the present treatment it is simpler to limit the discussion to total dry matter and annual rainfall.

#### *Relation between activity coefficient and water*

In this section an approximate expression for the activity coefficient in the modified Mitscherlich equation (8),  $\epsilon_{t,\theta}$ , is presented, which can be tested under field condi-

tions. To this end it is assumed that:

$$\epsilon_{t,\theta} = \epsilon_{nt} Y_{\theta}^n \quad (14)$$

where  $n$  is a power constant, which is independent of  $\theta$ . As the dimension of  $\epsilon_{t,\theta}$  is  $\text{kg dm kg}^{-1}$  nutrient, it follows that the dimension of  $\epsilon_{nt}$  is  $\text{kg}^{1-n} \text{ dm ha}^n \text{ kg}^{-1}$  nutrient, that is, the dimension of  $\epsilon_{nt}$  depends on the value of the power constant,  $n$ , in such a way that the dimension of the product  $\epsilon_{nt} Y_{\theta}^n$  equals  $\text{kg dm kg}^{-1}$  nutrient. Furthermore, the power constant,  $n$ , is expected to be larger than zero, that is,  $\epsilon_{t,\theta}$  is expected to be an increasing function of  $\theta$ .

It may be noted that  $Y_{\theta}$  by itself can be considered a measure of seasonal availability of moisture to crops at different locations in the agro-ecological zone under consideration. That is, the water-limited potential yield itself is the best estimate of available moisture, if other factors are not limiting crop growth. With the use of Equation (12), the above expression for  $\epsilon_{t,\theta}$  (Equation 14) would be a rather versatile function of rainfall, which could approximate almost any rainfall-dependency over a limited range of rainfall conditions.

With the use of Equation (14), Equation (8) can now be written as:

$$Y = Y_{\theta} - Y_{\theta} \exp(-\epsilon_{nt} N_t Y_{\theta}^{n-1}) \quad (15)$$

Equation (15) can be evaluated experimentally, once  $Y_{\theta}$  is known as a function of the seasonal availability of water (cf. Equation 12), in a well-defined agro-ecological environment.

### *Soil and fertilizer nutrients*

In Equation (3) the source and chemical nature of the nutrients are not specified. Nevertheless, it is quite likely that nutrient uptake and availability will also be affected by source (e.g., chemical fertilizer, organic residue, soil minerals) or chemical nature (inorganic or organic forms). Assuming that the crop derives its nutrients mainly from two sources, native soil nutrients and fertilizer-applied nutrients,  $N_t$  may be written as:

$$N_t = N_s + N_f \quad (16)$$

where  $N_s$  refers to soil nutrients ( $\text{kg nutrient ha}^{-1}$ ) in the rooting zone of the crop, and  $N_f$  refers to fertilizer-applied nutrients. Equation (16) could be expanded to include the effects of nutrients derived from other sources (e.g., atmospheric deposition, crop residues, biological N fixation), timing of fertilizer application (e.g., basal vs topdressing), type of application (e.g., banding or placement vs broadcast) or residual effects of fertilizer application in previous seasons. In the present treatment, however, the discussion will be limited to soil- and fertilizer-derived nutrients. The effect of moisture on nutrient availability can be introduced in Equation (16):



$$\varepsilon_{t,\theta}N_t = \varepsilon_{s,\theta}N_s + \varepsilon_{f,\theta}N_f \quad (17)$$

where the activity coefficients  $\varepsilon_{s,\theta}$  and  $\varepsilon_{f,\theta}$  are measures for the moisture-dependent availabilities of soil and fertilizer nutrients, respectively. Once fertilizer applied nutrients are fully mixed with nutrients in the soil pool, it is difficult to distinguish between the two sources of nutrients, other than through the use of isotopes. However, fertilizer nutrients may be subject to losses upon application, such as through volatilization (nitrogen) or surface runoff, which would result in lower availabilities or nutrient-use efficiencies of fertilizer-applied nutrients than of soil nutrients.

The soil nutrient pool,  $N_s$ , may be a function of  $\theta$  as well, for example, through increased mineralization of organically bound nutrients or increased dissolution of soil minerals under high rainfall conditions. As this effect is difficult to distinguish from an increased nutrient availability, resulting from increased root development and increased accessibility of soil nutrients under higher rainfall conditions, the effect of this is included in  $\varepsilon_{s,\theta}$ . Nevertheless, if information on the soil nutrient pool is available, this could be explicitized through the introduction of  $N_{s,\theta}$ .

Assuming that the crop may use soil and fertilizer nutrients with different efficiencies, the Mitscherlich equation (8) may thus be written:

$$Y = Y_\theta - Y_\theta \exp(-(\varepsilon_{s,\theta}N_s + \varepsilon_{f,\theta}N_f)/Y_\theta) \quad (18)$$

where  $Y_\theta$  has been set equal to zero, because yield is assumed to be zero if  $N_s = N_f = 0$ . The use of Equation (18) makes sense only if information on the availability and crop uptake of soil- and fertilizer-derived nutrients is available from field experiments. Under such conditions, Equation (18) can be written in the alternative form:

$$Y = Y_\theta - Y_\theta \exp(-(\varepsilon_{ns}N_s Y_\theta^{n-1} + \varepsilon_{mf}N_f Y_\theta^{m-1})) \quad (19)$$

where  $\varepsilon_{ns}$  and  $\varepsilon_{mf}$  are defined in a similar way to Equation (14) and where it has been assumed that the power constants  $n$  and  $m$  can be measured separately for different sources of nutrients. If this is not the case, or if they are equal, it follows that  $n = m$ .

#### *Relation between expected yield, required soil nutrient availability and seasonal rainfall*

In principle, a relation between the soil nutrient availability required to achieve a specified level of expected yield, as a function of seasonal rainfall, can be derived from Equation (19). For  $N_f = 0$ , Equation (19) can be written as:

$$N_s = \varepsilon_{ns}^{-1} Y_\theta^{1-n} \ln\{Y_\theta/(Y_\theta - Y)\} \quad (20)$$

Taking, as an example,  $Y = 0.8Y_\theta$ , that is, actual yield is expected to be 80% of water-limited potential yield, it follows that:

$$N_s = 1.61 \varepsilon_{ns}^{-1} Y_\theta^{1-n} \quad (21)$$

where the expected yield level, 80% of water-limited potential yield, has been selected more or less arbitrarily. However, in the semi-arid regions, farmers would rarely aim at 100% of  $Y_{\theta}$ , or even 90%, because of uncertainty about actual rainfall in a particular growing season. Assuming that Equation (12) applies, Equation (21) becomes:

$$N_s = 1.61 \varepsilon_{ns}^{-1} (Y_x \varepsilon_{\theta r} \Delta r)^{1-n} \quad (22)$$

which is a function of annual rainfall only. This relation is of significance for assessing the required availability of soil nutrients, in particular available phosphorus (e.g., P-Olsen), which is known to be affected by available moisture (Matar *et al.*, 1992).

## Discussion

The modified Mitscherlich equation, in the form of Equation (15), is a simple and straightforward way to introduce moisture-dependency in the Mitscherlich equation. There is little doubt that, ultimately, achievable maximum yields under rainfed conditions in the semi-arid regions of the world are limited by available moisture up to a certain threshold level where other factors may become limiting. As it is known that biomass production is zero below a certain threshold level of rainfall ( $r_0$ ) and then increases with increasing rainfall, water-limited potential yield of a particular crop can be described, in first approximation, by a linear function of rainfall (cf. Equation 12), for a limited range of agro-ecological and edaphic conditions.

Furthermore, the introduction of a simple and versatile function of moisture in the exponent of Equation (15) to account for the effect of seasonal water-availability on nutrient availability and nutrient-use efficiency seems plausible. For example, if there is no moisture in a certain layer of the soil (e.g., the top layer) the crop cannot use any of the nutrients contained in that layer, that is, they are inaccessible because of lack of water. Similarly, soil phosphorus is known to be less available to crops under dry conditions than at higher levels of available moisture (Matar *et al.*, 1992; Harmsen, 1995). At the other end of the spectrum, under high rainfall conditions, leaching of nitrate to beyond the reach of the rooting system of the crop or denitrification could effectively decrease the availability of soil mineral nitrogen.

The function  $Y_{\theta}^{n-1}$  is a simple and straightforward way of introducing moisture dependency in the exponent of the Mitscherlich equation. This function can approximate a range of, increasing or decreasing, relationships between nutrient availability and available moisture. Again, this function is not expected to be valid beyond a certain range of rainfall conditions.

In Figure 1, the modified Mitscherlich equation in the form of Equation (15), with grain yield, GY, replacing Y, is plotted for 3 values of  $GY_{\theta}$  and  $n-1 = -1$ . This represents essentially the Liebig-model and it can be seen that the slopes of all response curves are the same at  $N_t = 0$ . In Figure 2, the modified Mitscherlich equation (Equation 15) is plotted for  $n-1 = 0$  and for 3 values of  $GY_{\theta}$ . In the case of  $n-1 = 0$ , the

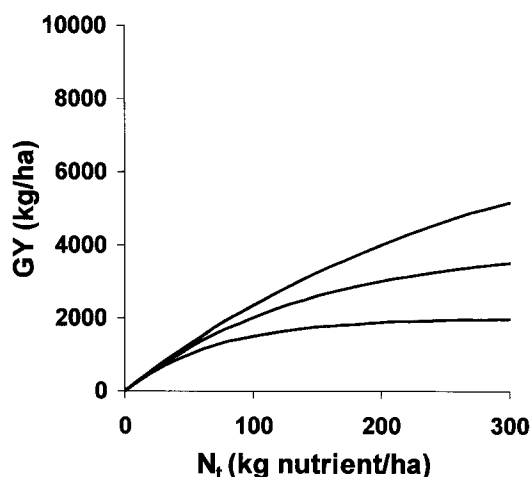


Figure 1. The modified Mitscherlich equation (Equation 15), where  $Y$  has been replaced by  $GY$  and  $Y_0$  by  $GY_0$ , is plotted for 3 values of water-limited potential grain yield,  $GY_0 = 2000, 4000$  and  $8000$  kg grain  $ha^{-1}$ , and for the power constant  $n-1 = -1$  (i.e.,  $n = 0$ ). The value of  $\epsilon_{nt}$  is taken as  $27.73$  kg grain  $kg^{-1}$  nutrient.

equation reduces to the classical Mitscherlich equation (2). It can be seen that the slopes of the response curves are all different at  $N_t = 0$ .

Crop nutrient uptake may be denoted by  $N_p$  (kg nutrient  $ha^{-1}$  year $^{-1}$ ) and nutrient uptake by the grain only may be denoted by  $GN_p$  (kg nutrient  $ha^{-1}$  year $^{-1}$ ). Assuming that the nitrogen content in the grain is 2%, the yield response curves can also be interpreted as nitrogen uptake ( $GN_p$ ) curves, e.g.,  $GN_p = 0.02GY$ . If it is further assumed that all crop-available nitrogen is used by the crop, i.e.,  $GN_p = N_t$ , it follows that at  $N_t = 100$  kg nitrogen  $ha^{-1}$ , the nitrogen uptake by the grain would be  $34.6$  ( $GY_0 = 2000$ ),  $69.2$  ( $GY_0 = 4000$ ) and  $138.3$  kg nitrogen  $ha^{-1}$  ( $GY_0 = 8000$  kg  $ha^{-1}$ ). The broken line in Figure 2 indicates where the nitrogen uptake by the crop would equal the total amount of nitrogen available ( $GN_p = N_t$ ). The area to the left of the broken line thus indicates where the nitrogen uptake exceeds the total amount of ni-

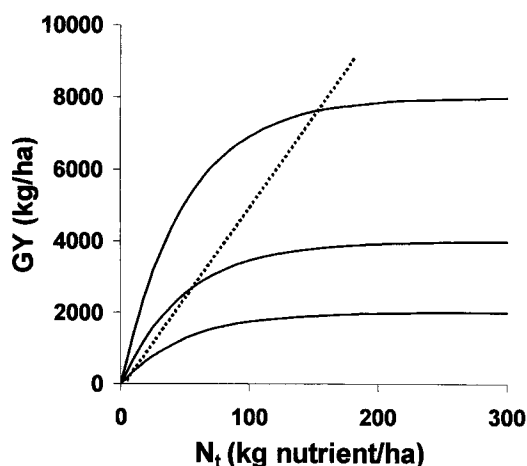


Figure 2. The modified Mitscherlich equation (Equation 15) for  $n-1 = 0$  (i.e.,  $n = 1$ ) is plotted for three values of water-limited potential grain yield  $GY_0 = 2000, 4000$  and  $8000$  kg grain  $ha^{-1}$ . The value of  $\epsilon_{nt}$  is set at  $0.020$  kg $^{-1}$  nutrient  $ha$ . The broken line represents the relationship  $GY = 50N_p$ , based on 2% N in the grain.

trogen available. It may be noted that actual nitrogen uptake would, in fact, be about 30–40% higher, as also the straw contains some nitrogen (e.g., about 0.5%). Also, in the semi-arid tropics, nitrogen use efficiencies of 40–60% could easily be achieved (Harmsen, 1984; Buresh *et al*, 1990), but 100% would be very rare.

Finally, the notion of ‘nutrient availability’ is a composite parameter, which is made up of a number of factors, such as root activity, spatial and temporal distribution of root distribution and density, moisture dynamics, gas exchange, and distribution and chemical form of nutrients in the soil. In the present approach, the dynamics of the soil nutrient pool, including chemical transformations, interactions with the solid phase, and immobilization-mineralization reactions, as well as the moisture dynamics, are effectively all included in the notion “availability” and expressed as a function of seasonal rainfall (cf. Equations 12 and 14). Obviously, this is a rather crude approach, but it allows the model to be fitted to yield response to nutrient application under field conditions.

The more elegant approach to including moisture and nutrient dynamics in a crop production model would be through a mechanistic description of soil water and nutrient dynamics in relation to crop development under specified climatic and soil conditions (simulation modelling). However, such approaches do require significant amounts of data, often on a daily or weekly basis, which are not always available from field experiments. When such data are not available, production functions of the Liebig- or Mitscherlich-Baule-type could be used instead to relate yield data to annual rainfall, fertilizer inputs, etc.

The present approach of modifying the Mitscherlich equation for rainfed agricultural production in semi-arid regions, to allow for the effects of variation in annual rainfall, is a step forward over the classical Mitscherlich equation. The present model has the further advantage that it requires a limited amount of data for calibration: i.e., total seasonal rainfall, yield (grain or dm), soil nutrient level (e.g., P-Olsen, mineral-N, exchangeable-K).

The modified Mitscherlich equation could, in principle, be used to assess fertilizer requirements based on expected water-limited potential yield, which, in turn, depends on expected rainfall. The latter element, of course, is always there: any sort of fertilizer recommendations for rainfed crop production in the semi-arid regions has to be based on some form of prediction or expectation of seasonal rainfall. Nevertheless, with the modified Mitscherlich model, nutrient requirements can be predicted, based on a simple soil test, for a specified yield level.

The model could also be used to determine whether a certain crop did achieve the expected yield level during a certain season, based on actual rainfall and nutrient conditions. If so, this would be an indication that no other factors limited (or reduced) yield during the season. If not, one would have to analyze the factors that could limit or reduce yield and determine the cause of the lower yield.

## Conclusion

In the classical Mitscherlich equation, the maximum, or potential, yield is assumed to be a constant. In the semi-arid regions, potential yields are determined by crop-

available moisture. Also, potential nutrient uptake and the availability of nutrients is affected by available moisture. The modified Mitscherlich equation presented in this paper, considers the effects of moisture on potential yield, potential nutrient uptake and nutrient availability in a simple and direct way. The model discussed in this paper needs further testing, but could provide a framework for a Mitscherlich-type model to describe crop response to nutrient availability c.q. application under rain-fed conditions in the semi-arid regions.

## References

- Baule, B., 1917. Mitscherlich's law of physiological relations (In German). *Landwirtschaftliche Jahrbuecher* 51 (1916–17): 363–385.
- Buresh, R.J., P.L.G. Vlek & K. Harmsen, 1990. Fate of fertilizer nitrogen applied to wheat under simulated Mediterranean environmental conditions. *Fertilizer Research* 23: 25–36.
- Cooper, P.J.M., 1983. Crop management in rainfed agriculture with special reference to water use efficiency. In: Proceedings of the 17th Colloquium of the International Potash Institute. International Potash Institute (IPI), Bern, pp. 63–79.
- Cooper, P.J.M., J.D.H. Keatinge & G. Hughes, 1983. Crop evapotranspiration – A technique for calculation of its components by field measurements. *Field Crops Research* 7: 299–312.
- De Vries, O., 1939. Yield curves and yield laws; the interaction of the growth factors (In Dutch). *Landbouwkundig Tijdschrift* 51(630): 727–763.
- De Wit, C.T., 1992. Resource use efficiency in agriculture. *Agricultural Systems* 40: 125–151.
- De Wit, C.T., 1994. Resource use analysis in agriculture: a struggle for interdisciplinarity. pp. 41–55. In: Louise O. Fresco, Leo Stroosnijder, Johan Bouma and Herman van Keulen (editors). *The Future of the Land: Mobilising and Integrating Knowledge for Land Use Options*. John Wiley & Sons, Chichester, England.
- Harmsen, K., 1984. Nitrogen fertilizer use in rainfed agriculture. *Fertilizer Research* 5: 371–382.
- Harmsen, K., 1995. Integrated phosphorus management: a modified Mitscherlich equation for predicting the response to phosphorus in dryland agriculture. In: R. Dudal and R.N. Roy (editors). *Integrated Plant Nutrition Systems. Report of an Expert Consultation. FAO Fertilizer and Plant Nutrition Bulletin* 12: 293–306.
- Harmsen, K., 2000. A modified Mitscherlich equation for rainfed crop production in semi-arid areas: 2. A case study of cereals in Syria. *Netherlands Journal of Agricultural Science* 48:251–272.
- Harmsen, K., K.D. Shepherd & A.Y. Allan, 1983. Crop response to nitrogen and phosphorus in rainfed agriculture. In: Nutrient Balances and the Need for Fertilizers in Semi-Arid and Arid Regions. *Proceedings of the 17th Colloquium of the International Potash Institute*. International Potash Institute (IPI), Bern, pp. 223–248.
- Liebscher, G., 1895. Study on the determination of the fertilizer needs of agricultural soils and crops (In German). *Journal fuer Landwirtschaft* 43: 49–216.
- Matar, A., J. Torrent & J. Ryan, 1992. Soil and fertilizer phosphorus and crop responses in the dryland Mediterranean zone. *Advances in Soil Science* 18: 81–146.
- Mitscherlich, E.A., 1909. The law of the minimum and the law of diminishing soil productivity (In German). *Landwirtschaftliche Jahrbuecher* 38: 537–552.
- Mitscherlich, E.A., 1913. Soil science for agriculture and forestry (In German). 2nd edition, 317 pp. Verlag Paul Parey, Berlin.
- Mitscherlich, E.A., 1956. Yield laws (In German). Akademie-Verlag, Berlin. 75 pp.
- Monteith, J.L., 1981. Epilogue: themes and variations. In: John Monteith & Colin Webb (editors), *Soil Water and Nitrogen in Mediterranean-type Environments*. Martinus Nijhoff Publishers, The Hague, The Netherlands. pp. 305–309. Also published as *Plant and Soil* 58 (1981).
- Theophrastus, 1989. *Enquiry into Plants (Historia Plantarum)*, Books VI–IX, translated by Arthur F. Hort, Loeb Classical Library, Volume 79 (1989) Book VIII, 7.6. Harvard University Press, Cambridge, Massachusetts.

- Theophrastus, 1990. *De Causis Plantarum*, Books III-IV, translated by Benedict Einarson and George K.K. Link, Loeb Classical Library, Volume 474 (1990) Book III, 23.4. Harvard University Press, Cambridge, Massachusetts.
- Van Der Paauw, F., 1938. On the relation between growth factors and yield, and the principles that determine this relationship (In Dutch). *Landbouwkundig Tijdschrift* 50: 795–830.
- Von Boguslawski, E., & B. Schneider, 1962. The third approximation of the yield law: 1st communication (In German). *Zeitschrift fuer Acker- und Pflanzenbau* 114(3): 221–236.
- Von Boguslawski, E., & B. Schneider, 1963. The third approximation of the yield law. 2nd communication (In German). *Zeitschrift fuer Acker- und Pflanzenbau* 116(2): 113–128.
- Von Liebig, J., 1855. *The basics of agricultural chemistry with reference to the experiments conducted in England* (In German). Vieweg und Sohn, Braunschweig, Germany.